

Quantum Mechanics I — Exercise List 1

PPGF / UERJ — 2026.1

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Instructions. Each list contains three levels of problems:

- **Level A:** direct application of definitions and results from the lectures.
- **Level B:** problems that require combining concepts or adapting methods to new situations.
- **Level C:** a challenging problem — it may involve a proof, a result from the research literature, or a question requiring deeper insight.

The estimated time is a rough guide for an average graduate student working individually. You are encouraged to discuss with colleagues, but each student must write their own solutions.

Principles and Formalism (Module I)

Topics: Hilbert space and Dirac notation; spectral representation; finite-dimensional spaces and matrix representations; uncertainty relations; harmonic oscillator; coherent and squeezed states.

Distribute: Lecture 5 (23/03). Evaluation: 06/04.

Level A

- 1A.1.** Consider a three-dimensional Hilbert space with orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle\}$. An operator $\hat{A}$ is represented in this basis by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

- Verify that $\hat{A}$ is Hermitian.
- Find the eigenvalues and normalized eigenstates of $\hat{A}$.
- Write the spectral representation $\hat{A} = \sum_n a_n |a_n\rangle\langle a_n|$ and verify it by explicit matrix multiplication.
- A system is prepared in the state $|\psi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$. What are the possible outcomes of a measurement of $\hat{A}$ and their probabilities? What is the state immediately after obtaining the eigenvalue -1 ?

(estimated time: 45 min)

Reference: Lecture notes §1, §1.2; Sakurai §1.3.

1A.2. For the one-dimensional harmonic oscillator, use the algebraic method ($\hat{a}$, $\hat{a}^\dagger$) to compute:

- (a) $\langle n|\hat{X}|m\rangle$ and $\langle n|\hat{P}|m\rangle$ for arbitrary n, m .
- (b) $\langle n|\hat{X}^2|n\rangle$ and $\langle n|\hat{P}^2|n\rangle$.
- (c) Verify the uncertainty relation $\langle(\Delta X)^2\rangle\langle(\Delta P)^2\rangle \geq \hbar^2/4$ for the state $|n\rangle$ and show that equality holds only for $n = 0$.

(estimated time: 40 min)

Reference: Lecture notes §1.4, eq. (100)–(101); Sakurai §2.3.

1A.3. Let $|z\rangle$ be a coherent state of the harmonic oscillator with $z = |z|e^{i\phi}$.

- (a) Compute $\langle z|\hat{N}|z\rangle$ and $\langle z|\hat{N}^2|z\rangle$, where $\hat{N} = \hat{a}^\dagger\hat{a}$.
- (b) Show that the variance of the number operator is $(\Delta N)^2 = |z|^2$ and therefore the relative fluctuation $\Delta N/\langle N\rangle = 1/|z|$ decreases for large $|z|$.
- (c) What is the probability of finding n quanta in the coherent state? Identify the distribution.

(estimated time: 30 min)

Reference: Lecture notes §1.5, eqs. (126), (134)–(135).

Level B

1B.1. Minimum uncertainty states. A normalized state $|\psi\rangle$ saturates the Robertson uncertainty relation for two observables $\hat{A}$ and $\hat{B}$ when

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle = \frac{1}{4}|\langle[\hat{A}, \hat{B}]\rangle|^2$$

- (a) Show that this condition is equivalent to $(\hat{A} - \langle A\rangle + i\gamma(\hat{B} - \langle B\rangle))|\psi\rangle = 0$ for some real constant γ .
- (b) Specialize to $\hat{A} = \hat{X}$, $\hat{B} = \hat{P}$. Show that the minimum uncertainty states are Gaussian wave packets and determine the most general form of $\psi(x)$.
- (c) Identify the coherent states and squeezed states of the harmonic oscillator within this family by relating γ to the squeezing parameter r .

(estimated time: 60 min)

Reference: Lecture notes §1.2 (eq. 53–54), §1.5; Sakurai §1.4.

1B.2. Displacement operator algebra.

- (a) Show that $D(\alpha)D(\beta) = D(\alpha + \beta)e^{i\text{Im}(\alpha\beta^*)}$.
- (b) Using (a), show that the coherent states satisfy $\langle\beta|\alpha\rangle = \exp(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \beta^*\alpha)$.
- (c) Define the Wigner function of a state $\hat{\rho}$ as

$$W(\alpha) = \frac{2}{\pi}\text{Tr} \left[\hat{\rho} D(\alpha) (-1)^{\hat{N}} D^\dagger(\alpha) \right]$$

Compute $W(\alpha)$ for a coherent state $|\beta\rangle$ and for the Fock state $|n\rangle$ with $n = 0$ and $n = 1$.

(estimated time: 75 min)

Reference: Lecture notes §1.5, eq. (180); Ballentine §19.3.

Level C

1C.1. The GNS construction for a qubit. Consider the algebra $\mathcal{A} = M_2(\mathbb{C})$ (the algebra of 2×2 complex matrices) and the state

$$\omega(A) = \text{Tr}(\rho A), \quad \rho = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}, \quad 0 < p < 1$$

- (a) Compute the Gelfand ideal $\mathcal{J} = \{X \in \mathcal{A} : \omega(X^\dagger X) = 0\}$ and show that it is trivial ($\mathcal{J} = \{0\}$) for $0 < p < 1$.
- (b) Construct the GNS Hilbert space $\mathcal{H}_\omega$ explicitly. What is its dimension? Identify the cyclic vector $|\Omega\rangle$.
- (c) Construct the representation $\pi_\omega(A)$ for the Pauli matrices and verify that $\omega(\sigma_z) = \langle \Omega | \pi_\omega(\sigma_z) | \Omega \rangle$.
- (d) Show that for $p = 1$ (a pure state), $\mathcal{J}$ is nontrivial and the GNS Hilbert space has dimension 2 instead of 4. Interpret this result.

(estimated time: 90 min)

Reference: Lecture notes §1.1, eqs. (19)–(24); Ballentine §2.4.
