

Quantum Mechanics I — Exercise List 2

PPGF / UERJ — 2026.1

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Instructions. Each list contains three levels of problems:

- **Level A:** direct application of definitions and results from the lectures.
- **Level B:** problems that require combining concepts or adapting methods to new situations.
- **Level C:** a challenging problem — it may involve a proof, a result from the research literature, or a question requiring deeper insight.

The estimated time is a rough guide for an average graduate student working individually. You are encouraged to discuss with colleagues, but each student must write their own solutions.

Quantum Dynamics (Module II)

Topics: Time evolution operator; Schrödinger and Heisenberg pictures; neutrino-like oscillations; dynamics of the harmonic oscillator and coherent states; propagator; gauge transformations; Aharonov-Bohm effect.

Distribute: Lecture 10 (13/04). Evaluation: 22/04.

Level A

2A.1. Two-level oscillations. A system has two energy eigenstates $|E_1\rangle, |E_2\rangle$ with $E_2 > E_1$. An observable $\hat{A}$ has eigenstates related to the energy eigenstates by

$$|A_+\rangle = \cos\theta |E_1\rangle + \sin\theta |E_2\rangle, \quad |A_-\rangle = -\sin\theta |E_1\rangle + \cos\theta |E_2\rangle$$

At $t = 0$ the system is in the state $|A_+\rangle$.

- Calculate the probability $P_{+\rightarrow-}(t)$ of finding the system in $|A_-\rangle$ at time t .
- For what values of θ is the oscillation maximal? Interpret.
- Estimate the oscillation frequency for a neutrino with momentum $p \gg m_i c$ (where m_i are the mass eigenvalues), showing that it is proportional to $\Delta m^2 \equiv m_2^2 - m_1^2$.

(estimated time: 40 min)

Reference: Lecture notes §1.6, eq. (158)–(160); Sakurai §2.1.

2A.2. Heisenberg picture for the harmonic oscillator.

- Starting from the Heisenberg equations of motion for $\hat{a}(t)$ and $\hat{a}^\dagger(t)$, derive $\hat{X}(t)$ and $\hat{P}(t)$ as given in the lecture notes, eq. (177).

- (b) Compute $\langle n|\hat{X}(t)|n\rangle$ and $\langle n|\hat{X}(t)^2|n\rangle$ in the Heisenberg picture and verify that the result is time-independent (as it must be for an energy eigenstate).
- (c) Now compute $\langle z|\hat{X}(t)|z\rangle$ for a coherent state $|z\rangle$ (with $z = |z|e^{i\phi}$) and show that it oscillates as a classical harmonic oscillator.

(estimated time: 45 min)

Reference: Lecture notes §1.6–§1.7, eqs. (173)–(177); Sakurai §2.3.

Level B

2B.1. Free-particle propagator. The propagator (Feynman kernel) is defined as $K(\mathbf{x}, t; \mathbf{x}', 0) = \langle \mathbf{x}|e^{-i\hat{H}t/\hbar}|\mathbf{x}'\rangle$.

- (a) For a free particle in one dimension ($\hat{H} = \hat{P}^2/2m$), compute the propagator by inserting a complete set of momentum eigenstates. Show that

$$K(x, t; x', 0) = \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left(\frac{im(x - x')^2}{2\hbar t}\right)$$

- (b) Verify that K satisfies the Schrödinger equation in the variables (x, t) .
- (c) Show that $K(x, t; x', 0) \rightarrow \delta(x - x')$ as $t \rightarrow 0^+$.
- (d) A particle starts in the Gaussian wave packet $\psi(x', 0) = (\pi\sigma^2)^{-1/4} \exp(-x'^2/2\sigma^2)$. Use the propagator to find $\psi(x, t)$ and compute the time-dependent width $\Delta x(t)$.

(estimated time: 60 min)

Reference: Sakurai §2.6; Cohen-Tannoudji vol. 1, Complement G_{III} .

2B.2. Aharonov-Bohm effect. Consider a charged particle confined to move on a ring of radius R in the xy -plane, encircling an infinite solenoid along the z -axis carrying magnetic flux Φ . The vector potential outside the solenoid is $\mathbf{A} = \frac{\Phi}{2\pi R}\hat{\phi}$ (in cylindrical coordinates).

- (a) Write the Hamiltonian for the particle on the ring in terms of the angular variable ϕ and the canonical angular momentum $\hat{L}_z = -i\hbar\partial_\phi$.
- (b) Find the energy eigenvalues as a function of the flux Φ . Show that they depend on Φ even though $\mathbf{B} = 0$ at the position of the particle.
- (c) Show that the spectrum is periodic in Φ with period $\Phi_0 = hc/e$ (the flux quantum). Interpret.
- (d) Discuss what this implies about the role of the electromagnetic potential versus the electromagnetic field in quantum mechanics.

(estimated time: 50 min)

Reference: Sakurai §2.7; Ballentine §11.5.

2B.3. Ehrenfest's theorem.

- (a) Starting from Heisenberg's equation, derive the Ehrenfest relations:

$$m\frac{d}{dt}\langle\hat{\mathbf{X}}\rangle = \langle\hat{\mathbf{P}}\rangle, \quad \frac{d}{dt}\langle\hat{\mathbf{P}}\rangle = -\langle\nabla V(\hat{\mathbf{X}})\rangle$$

- (b) Under what conditions do these reduce to Newton's equations for the expectation values? (Hint: consider when $\langle \nabla V(\hat{\mathbf{X}}) \rangle \approx \nabla V(\langle \hat{\mathbf{X}} \rangle)$.)
- (c) Verify Ehrenfest's theorem explicitly for the coherent state $|z\rangle$ of the harmonic oscillator by using the results from problem 2A.2 and the lecture notes.

(estimated time: 40 min)

Reference: Sakurai §2.2; Ballentine §4.3.

Level C

2C.1. Coherent states under a forced oscillator. The lecture notes (§1.7) show that a coherent state remains coherent under the Hamiltonian $\hat{H}(t) = \hbar[\omega(t)\hat{a}^\dagger\hat{a} + f(t)\hat{a}^\dagger + f^*(t)\hat{a} + g(t)]$, with $|\psi(t)\rangle = e^{i\theta(t)}|z(t)\rangle$.

- (a) For the specific case $\omega(t) = \omega_0$ (constant), $f(t) = f_0 e^{-i\Omega t}$ (monochromatic driving), and $g = 0$, solve the equation $i\dot{z} = \omega_0 z + f_0 e^{-i\Omega t}$ with initial condition $z(0) = 0$.
- (b) Calculate $|z(t)|^2$ (the mean number of quanta) as a function of time. Show that it exhibits resonance when $\Omega \rightarrow \omega_0$ and determine the behavior at exact resonance.
- (c) Compute $\theta(t)$ and discuss whether the phase θ contributes to any observable quantity. (Hint: consider the expectation value of $\hat{X}(t)$.)
- (d) Discuss the connection with the classical forced harmonic oscillator. In what precise sense does the coherent state follow the classical trajectory?

(estimated time: 90 min)

Reference: Lecture notes §1.7; Sakurai §2.3 (problem 2.22 in 3rd ed.).
