

Quantum Mechanics I — Exercise List 3

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Instructions. Each list contains three levels of problems:

- **Level A:** direct application of definitions and results from the lectures.
- **Level B:** problems that require combining concepts or adapting methods to new situations.
- **Level C:** a challenging problem — it may involve a proof, a result from the research literature, or a question requiring deeper insight.

The estimated time is a rough guide for an average graduate student working individually. You are encouraged to discuss with colleagues, but each student must write their own solutions.

Open Quantum Systems and Quantum Information (Module III)

Topics: Density operator; partial trace; qubits and Bloch sphere; entanglement and Bell states; Bell/CHSH inequality; von Neumann entropy; quantum channels; teleportation and no-cloning.

Distribute: Lecture 15 (04/05). Evaluation: 13/05.

Level A

3A.1. Density matrix for a qubit. The most general density operator for a single qubit can be written as

$$\hat{\rho} = \frac{1}{2}(\mathbf{1} + \mathbf{r} \cdot \boldsymbol{\sigma})$$

where $\mathbf{r} = (r_x, r_y, r_z)$ is the Bloch vector.

- Show that $\hat{\rho}$ is Hermitian with unit trace for any real $\mathbf{r}$.
- Show that $\hat{\rho} \geq 0$ (nonnegative eigenvalues) if and only if $|\mathbf{r}| \leq 1$.
- Show that $\hat{\rho}$ describes a pure state if and only if $|\mathbf{r}| = 1$. What is the Bloch vector for the states $|0\rangle$, $|1\rangle$, and $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$?
- Compute the von Neumann entropy $S(\hat{\rho})$ as a function of $|\mathbf{r}|$ and sketch the result.

(estimated time: 45 min)

Reference: Lecture notes §2 (Example: Q-bits); Nielsen & Chuang §2.4.

3A.2. Partial trace. Consider the two-qubit state $|\Psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$ (with $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$).

- (a) Compute the reduced density matrix $\hat{\rho}_A = \text{Tr}_B(|\Psi\rangle\langle\Psi|)$ by performing the partial trace explicitly.
- (b) Specialize to the Bell state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Show that $\hat{\rho}_A = \frac{1}{2}\mathbb{I}$.
- (c) Compute the von Neumann entropy of $\hat{\rho}_A$ for $|\Psi^-\rangle$ and interpret the result in terms of entanglement.
- (d) For a general product state $|\Psi\rangle = |\psi\rangle_A \otimes |\phi\rangle_B$, show that $\hat{\rho}_A = |\psi\rangle\langle\psi|$ (a pure state) and $S(\hat{\rho}_A) = 0$.

(estimated time: 40 min)

Reference: Lecture notes §2, eqs. (200)–(202); Nielsen & Chuang §2.4.

Level B

3B.1. The CHSH inequality. Consider two parties, Alice and Bob, each performing one of two possible measurements on their respective qubits. Alice measures either $\hat{A}_1$ or $\hat{A}_2$; Bob measures either $\hat{B}_1$ or $\hat{B}_2$. Define the CHSH operator

$$\hat{\mathcal{B}} = \hat{A}_1 \otimes (\hat{B}_1 + \hat{B}_2) + \hat{A}_2 \otimes (\hat{B}_1 - \hat{B}_2)$$

where each observable has eigenvalues ± 1 .

- (a) Show that any local hidden variable theory predicts $|\langle\mathcal{B}\rangle| \leq 2$.
- (b) Choose $\hat{A}_1 = \sigma_z$, $\hat{A}_2 = \sigma_x$, $\hat{B}_1 = \frac{1}{\sqrt{2}}(\sigma_z + \sigma_x)$, $\hat{B}_2 = \frac{1}{\sqrt{2}}(\sigma_z - \sigma_x)$. Compute $\langle\Psi^-|\hat{\mathcal{B}}|\Psi^-\rangle$ and show that it equals $-2\sqrt{2}$, violating the CHSH bound.
- (c) The value $2\sqrt{2}$ is the maximum quantum violation (Tsirelson's bound). Prove that $\hat{\mathcal{B}}^2 \leq 4\mathbb{I}$ and use this to show $|\langle\mathcal{B}\rangle| \leq 2\sqrt{2}$.

(estimated time: 60 min)

Reference: Nielsen & Chuang §2.6; Preskill, Lecture Notes, Ch. 4.

3B.2. Quantum channels: depolarizing and amplitude damping.

- (a) The *depolarizing channel* is defined by

$$\mathcal{E}(\hat{\rho}) = (1-p)\hat{\rho} + \frac{p}{3}(\sigma_x\hat{\rho}\sigma_x + \sigma_y\hat{\rho}\sigma_y + \sigma_z\hat{\rho}\sigma_z)$$

Show that this can be rewritten as $\mathcal{E}(\hat{\rho}) = (1 - \frac{4p}{3})\hat{\rho} + \frac{4p}{3}\frac{\mathbb{I}}{2}$. Verify that $\mathcal{E}$ maps the Bloch vector $\mathbf{r}$ to $(1 - \frac{4p}{3})\mathbf{r}$. For what value of p does the channel output the maximally mixed state regardless of input?

- (b) The *amplitude damping channel* has Kraus operators

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

Verify that $M_0^\dagger M_0 + M_1^\dagger M_1 = \mathbb{I}$. Apply this channel to a general qubit state $\hat{\rho} = \frac{1}{2}(\mathbb{I} + \mathbf{r} \cdot \boldsymbol{\sigma})$ and find the output Bloch vector $\mathbf{r}'$ as a function of $\mathbf{r}$ and γ .

- (c) Interpret the amplitude damping channel physically as spontaneous emission: if the qubit is an atom with ground state $|0\rangle$ and excited state $|1\rangle$, explain the role of γ . What is the fixed point of the channel (the state $\hat{\rho}_*$ such that $\mathcal{E}(\hat{\rho}_*) = \hat{\rho}_*$)?

(estimated time: 60 min)

Reference: Lecture notes §2 (quantum channels); Nielsen & Chuang §8.3.

3B.3. Quantum teleportation. Alice and Bob share the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AB}$. Alice has an additional qubit C in an unknown state $|\chi\rangle_C = \alpha|0\rangle + \beta|1\rangle$.

- Write the total state $|\chi\rangle_C \otimes |\Phi^+\rangle_{AB}$ and rewrite it in the Bell basis for the pair (C, A) .
- Alice performs a measurement in the Bell basis on qubits (C, A) . For each of the four outcomes, determine the state of Bob's qubit B .
- Show that Bob can recover $|\chi\rangle$ by applying one of $\{\mathbb{I}, \sigma_x, \sigma_y, \sigma_z\}$ depending on Alice's measurement result (communicated classically).
- Explain why teleportation does not violate the no-signaling principle (information cannot travel faster than light).

(estimated time: 50 min)

Reference: Nielsen & Chuang §1.3.7; Preskill, Lecture Notes, Ch. 4.

Level C

3C.1. Subadditivity of von Neumann entropy. Let $\hat{\rho}_{AB}$ be a density operator on a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$, with reduced states $\hat{\rho}_A = \text{Tr}_B(\hat{\rho}_{AB})$ and $\hat{\rho}_B = \text{Tr}_A(\hat{\rho}_{AB})$.

- Prove that $S(\hat{\rho}_{AB}) \leq S(\hat{\rho}_A) + S(\hat{\rho}_B)$, with equality if and only if $\hat{\rho}_{AB} = \hat{\rho}_A \otimes \hat{\rho}_B$.
Hint: Use Klein's inequality: $\text{Tr}(\hat{\rho} \ln \hat{\rho} - \hat{\rho} \ln \hat{\sigma}) \geq 0$ for any density operators $\hat{\rho}, \hat{\sigma}$, with equality iff $\hat{\rho} = \hat{\sigma}$. Apply it with $\hat{\sigma} = \hat{\rho}_A \otimes \hat{\rho}_B$.
- Prove the *triangle inequality* (Araki-Lieb, 1970):

$$|S(\hat{\rho}_A) - S(\hat{\rho}_B)| \leq S(\hat{\rho}_{AB})$$

Hint: Purify $\hat{\rho}_{AB}$ by introducing an auxiliary system R such that the total state $|\Psi\rangle_{ABR}$ is pure, then use the fact that for a pure state $S(\hat{\rho}_{AB}) = S(\hat{\rho}_R)$ and apply subadditivity to the partition $B|AR$.

- Show that for a pure state $|\Psi\rangle_{AB}$, $S(\hat{\rho}_A) = S(\hat{\rho}_B)$ (the subsystems have equal entanglement entropy). Is this still true for mixed states?

(estimated time: 90 min)

Reference: Lecture notes §2.1, eq. (210); Wilde, Quantum Information Theory, 2nd ed., §11.1–§11.3; Nielsen & Chuang §11.3.