

Quantum Mechanics I — Exercise List 4

PPGF / UERJ — 2026.1

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Instructions. Each list contains three levels of problems:

- **Level A:** direct application of definitions and results from the lectures.
- **Level B:** problems that require combining concepts or adapting methods to new situations.
- **Level C:** a challenging problem — it may involve a proof, a result from the research literature, or a question requiring deeper insight.

The estimated time is a rough guide for an average graduate student working individually. You are encouraged to discuss with colleagues, but each student must write their own solutions.

Symmetries and Angular Momentum (Module IV)

Topics: Translations and rotations as symmetry operations; angular momentum algebra and representations; spin-1/2; spherical harmonics; addition of angular momenta; Clebsch-Gordan coefficients; Wigner-Eckart theorem; parity and time reversal.

Distribute: Lecture 20 (03/06). *Evaluation:* 10/06.

Level A

4A.1. Spin-1/2 rotations.

- A spin-1/2 particle is in the state $|0\rangle$ ($\hat{S}_z$ eigenstate with eigenvalue $+\hbar/2$). It is rotated by an angle θ about the y -axis. Write the resulting state in the $\{|0\rangle, |1\rangle\}$ basis and verify that for $\theta = \pi/2$ one obtains the state $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ (up to phase).
- Compute the expectation values $\langle \hat{S}_x \rangle$, $\langle \hat{S}_y \rangle$, $\langle \hat{S}_z \rangle$ in the rotated state as functions of θ and verify that they transform as components of a vector under rotation.
- Show explicitly that a rotation by 2π about any axis gives $|\psi\rangle \rightarrow -|\psi\rangle$ for spin-1/2.

(estimated time: 35 min)

Reference: Lecture notes §4.6–§4.7; Sakurai §3.2–§3.3.

4A.2. Angular momentum matrices for $j = 1$.

- Using $\hat{J}_{\pm}|j, m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}|j, m \pm 1\rangle$, construct the 3×3 matrices for $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$ in the basis $\{|1, 1\rangle, |1, 0\rangle, |1, -1\rangle\}$.

- (b) Verify that $[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$ by explicit matrix multiplication.
- (c) Compute $\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$ and verify that it equals $2\hbar^2\mathbb{1}$, consistent with $j(j+1) = 2$.
- (d) Find the eigenvalues and eigenstates of $\hat{J}_x$ for $j = 1$.

(estimated time: 45 min)

Reference: Lecture notes §4.5; Sakurai §3.5.

4A.3. Clebsch-Gordan decomposition: $1 \otimes 1$.

- (a) Two particles have angular momentum $j_1 = 1$ and $j_2 = 1$. What are the allowed values of the total angular momentum J ? How many states are there in each J sector? Verify the total count.
- (b) Starting from the stretched state $|J = 2, M = 2\rangle = |1, 1\rangle|1, 1\rangle$, apply $\hat{J}_-$ to construct $|2, 1\rangle$ and $|2, 0\rangle$.
- (c) Construct $|1, 1\rangle$ by requiring orthogonality to $|2, 1\rangle$ within the $M = 1$ subspace. Verify normalization.

(estimated time: 50 min)

Reference: Lecture notes §5.1; Sakurai §3.8.

Level B

4B.1. Selection rules from the Wigner-Eckart theorem.

Consider the electric dipole operator $\hat{\mathbf{d}} = e\hat{\mathbf{X}}$ for an atom.

- (a) Write the spherical components $\hat{d}_0, \hat{d}_{\pm 1}$ of the dipole operator as a rank-1 spherical tensor.
- (b) Using the Wigner-Eckart theorem, determine the selection rules for the matrix element $\langle n', \ell', m' | \hat{d}_q | n, \ell, m \rangle$ (allowed values of $\Delta\ell$ and Δm).
- (c) Explain why the parity selection rule $\Delta\ell = \pm 1$ (odd) is consistent with the Wigner-Eckart rule $|\Delta\ell| \leq 1$.
- (d) The magnetic dipole operator $\hat{\boldsymbol{\mu}} \propto \hat{\mathbf{L}}$ is also a rank-1 tensor but is an axial vector (even parity). What are the selection rules for magnetic dipole transitions?

(estimated time: 50 min)

Reference: Lecture notes §5.2–§5.3; Sakurai §3.11.

4B.2. Time reversal of a spin-1/2 system.

- (a) Verify that $\hat{\Theta} = -i\sigma_y K$ (where K is complex conjugation in the $\{|0\rangle, |1\rangle\}$ basis) satisfies $\hat{\Theta}\hat{\mathbf{S}}\hat{\Theta}^{-1} = -\hat{\mathbf{S}}$.
- (b) Compute $\hat{\Theta}|0\rangle$ and $\hat{\Theta}|1\rangle$. Show that $\hat{\Theta}^2 = -\mathbb{1}$.
- (c) Consider the spin state $|\hat{n}, +\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$. Compute $\hat{\Theta}|\hat{n}, +\rangle$ and show that it is proportional to $|\hat{n}, -\rangle$ (the spin-down state along $\hat{n}$). Interpret: time reversal flips the spin direction.

- (d) Use Kramers' theorem to argue that the energy levels of an atom with an odd number of electrons must be at least doubly degenerate in the absence of a magnetic field. Give a physical example.

(estimated time: 50 min)

Reference: Lecture notes §5.4; Sakurai §4.4.

Level C

4C.1. The singlet as a decoherence-free state. Consider two qubits subjected to *collective dephasing*: the environment applies the same random rotation $e^{-i\phi\sigma_z/2} \otimes e^{-i\phi\sigma_z/2}$ to both qubits, where ϕ is a random variable uniformly distributed in $[0, 2\pi)$. The resulting quantum channel is

$$\mathcal{E}(\hat{\rho}) = \frac{1}{2\pi} \int_0^{2\pi} d\phi (U_\phi \otimes U_\phi) \hat{\rho} (U_\phi \otimes U_\phi)^\dagger, \quad U_\phi = e^{-i\phi\sigma_z/2}$$

- Show that the singlet state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is invariant under $U_\phi \otimes U_\phi$ for all ϕ : $(U_\phi \otimes U_\phi)|\Psi^-\rangle = |\Psi^-\rangle$.
- Compute $\mathcal{E}(|\Psi^-\rangle\langle\Psi^-|)$ and verify that it equals $|\Psi^-\rangle\langle\Psi^-|$ (no decoherence).
- Compute $\mathcal{E}(|\Phi^+\rangle\langle\Phi^+|)$ where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Show that the off-diagonal coherences are destroyed by the channel.
- Generalize: show that the singlet state is invariant under $U \otimes U$ for *any* $U \in SU(2)$, and explain why this makes the singlet subspace a “decoherence-free subspace” for any collective noise model. Relate this to the fact that the singlet carries the trivial representation $J = 0$ in the decomposition $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$.

(estimated time: 90 min)

Reference: Lecture notes §3.2 (Bell states), §5.5 (symmetries and quantum information); Preskill, Lecture Notes, Ch. 7; Zanardi & Rasetti, *Phys. Rev. Lett.* **79**, 3306 (1997).