

Quantum Mechanics I — Exercise List 5

PPGF / UERJ — 2026.1

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Instructions. Each list contains three levels of problems:

- **Level A:** direct application of definitions and results from the lectures.
- **Level B:** problems that require combining concepts or adapting methods to new situations.
- **Level C:** a challenging problem — it may involve a proof, a result from the research literature, or a question requiring deeper insight.

The estimated time is a rough guide for an average graduate student working individually. You are encouraged to discuss with colleagues, but each student must write their own solutions.

Approximation Methods and Scattering (Modules V–VI)

Topics: Time-independent perturbation theory (non-degenerate and degenerate); fine structure and Zeeman effect; variational method; time-dependent perturbation theory; Fermi's golden rule; Lippmann-Schwinger equation; Born approximation; partial waves; optical theorem.

Distribute: Lecture 24 (24/06). Evaluation: 08/07.

Level A

5A.1. Perturbation theory: basics. A one-dimensional harmonic oscillator has the perturbed Hamiltonian $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2) + \lambda\hbar\omega(\hat{a} + \hat{a}^\dagger)$.

- Compute the first-order energy correction $E_n^{(1)}$ for all n and show it vanishes.
- Compute the second-order energy correction $E_n^{(2)}$.
- This Hamiltonian can be solved *exactly* by completing the square (shifting the oscillator equilibrium). Find the exact eigenvalues and verify that the perturbation series truncates at second order.

(estimated time: 40 min)

Reference: Lecture notes §6.1; Sakurai §5.1.

5A.2. Born approximation for a Gaussian potential. Consider the potential $V(r) = V_0 e^{-r^2/a^2}$ (a Gaussian well of depth V_0 and range a).

- (a) Using the Born approximation formula for central potentials, compute the scattering amplitude $f^{(1)}(\theta)$. You will need the Fourier transform of a Gaussian.
- (b) Find the differential cross section $d\sigma/d\Omega$ and the total cross section σ_{tot} in the Born approximation.
- (c) Show that in the low-energy limit ($ka \ll 1$) the cross section becomes isotropic and compute the scattering length.
- (d) Determine the condition on V_0 , a , and the energy E for the Born approximation to be valid.

(estimated time: 45 min)

Reference: Lecture notes §7.3; Sakurai §7.2.

Level B

5B.1. Stark effect of the $n = 2$ level of hydrogen. The $n = 2$ level of hydrogen is 4-fold degenerate: $|2, 0, 0\rangle$ and $|2, 1, m\rangle$ ($m = -1, 0, 1$), where the quantum numbers are (n, ℓ, m) . A uniform electric field $\mathcal{E} = \mathcal{E}\hat{z}$ is applied.

- (a) Using the parity and Δm selection rules, show that the only nonzero matrix element of $\hat{V} = e\mathcal{E}\hat{z}$ within the $n = 2$ subspace is $\langle 2, 0, 0 | \hat{V} | 2, 1, 0 \rangle$ (and its conjugate).
- (b) Evaluate this matrix element. You will need the hydrogen wave functions:

$$\psi_{200} = \frac{1}{4\sqrt{2\pi}} \frac{1}{a_0^{3/2}} \left(2 - \frac{r}{a_0} \right) e^{-r/2a_0}, \quad \psi_{210} = \frac{1}{4\sqrt{2\pi}} \frac{1}{a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \cos \theta$$

Show that $\langle 2, 0, 0 | e\mathcal{E}z | 2, 1, 0 \rangle = -3e\mathcal{E}a_0$.

- (c) Diagonalize the perturbation matrix within the degenerate subspace and find the first-order energy corrections. How many distinct energy levels does the $n = 2$ multiplet split into?
- (d) This is the *linear* Stark effect ($E^{(1)} \propto \mathcal{E}$), in contrast to the quadratic Stark effect of the ground state. Explain why the $n = 1$ level has no linear Stark effect.

(estimated time: 60 min)

Reference: Lecture notes §6.2; Sakurai §5.2; Griffiths §7.3.

5B.2. Fermi's golden rule for a delta-function perturbation in time. A harmonic oscillator is in the ground state $|0\rangle$ when a brief perturbation $\hat{V}(t) = A\hat{x}\delta(t)$ is applied (a sudden “kick”).

- (a) Using first-order time-dependent perturbation theory, compute the transition probability $P_{0 \rightarrow n}$ for arbitrary n .
- (b) Show that the only allowed transition is $0 \rightarrow 1$ and compute $P_{0 \rightarrow 1}$ in terms of A , m , ω , and $\hbar$.
- (c) Verify that $\sum_n P_{0 \rightarrow n} \leq 1$ (to first order), and discuss under what condition on A this result is reliable.

- (d) Show that the same result can be obtained by treating the kick as an instantaneous displacement $D(\alpha)$ with $\alpha = iA/\sqrt{2m\hbar\omega}$, expanding the resulting coherent state in Fock states. Compare the exact result with the perturbative one.

(estimated time: 50 min)

Reference: Lecture notes §6.5–§6.6; Sakurai §6.1.

5B.3. Partial waves for a spherical well. Consider a particle scattering off a spherical potential well:

$$V(r) = \begin{cases} -V_0 & r < R \\ 0 & r > R \end{cases} \quad (V_0 > 0)$$

- (a) For s -wave ($\ell = 0$) scattering, solve the radial Schrödinger equation inside and outside the well. Apply the boundary conditions at $r = R$ (continuity of $u_0(r)$ and $u'_0(r)$, where $u_0 = rR_0$) to derive an expression for $\tan \delta_0$ in terms of k , R , and $K = \sqrt{2m(E + V_0)}/\hbar$.
- (b) Show that in the low-energy limit ($kR \ll 1$), the scattering length is $a_s = R \left(1 - \frac{\tan(KR)}{KR}\right)$.
- (c) The scattering length diverges when $\tan(KR) \rightarrow \infty$, i.e., $KR = (n + 1/2)\pi$. This corresponds to the appearance of a new bound state. Give a physical interpretation (hint: Levinson's theorem relates the number of bound states to $\delta_0(k = 0)$).

(estimated time: 60 min)

Reference: Lecture notes §7.4; Sakurai §7.6; Ballentine §13.4.

Level C

5C.1. The optical theorem from unitarity.

In this problem you will prove the optical theorem directly from the unitarity of the S -matrix, without using the partial wave expansion.

- (a) Starting from $\hat{S}^\dagger \hat{S} = \mathbb{1}$ and the decomposition $\hat{S} = \mathbb{1} + i\hat{R}$, show that $i(\hat{R}^\dagger - \hat{R}) = \hat{R}^\dagger \hat{R}$.
- (b) Taking the matrix element $\langle \mathbf{k} | \cdots | \mathbf{k} \rangle$ on both sides (with elastic scattering, $|\mathbf{k}'| = |\mathbf{k}|$), and using the relation between $\hat{R}$ and the scattering amplitude f , show that

$$2 \operatorname{Im} f(\theta = 0) = \frac{k}{2\pi} \int d\Omega' |f(\theta', \phi')|^2$$

- (c) Recognize the right-hand side as $\frac{k}{2\pi} \sigma_{\text{tot}}$ and thus derive $\sigma_{\text{tot}} = \frac{4\pi}{k} \operatorname{Im} f(0)$.
- (d) Discuss the physical meaning: why does the *imaginary* part of the forward amplitude determine the total cross section? What role does interference between the incident and scattered waves play?

- (e) Generalize: if inelastic channels are present, define the total cross section as $\sigma_{\text{tot}} = \sigma_{\text{el}} + \sigma_{\text{inel}}$ and show that the optical theorem still holds (the forward *elastic* amplitude determines the total cross section including inelastic processes).

(estimated time: 90 min)

Reference: Lecture notes §7.5–§7.6; Sakurai §7.3; Taylor, Scattering Theory, Ch. 8.